Stochastic distribution of compressive strength: effects on the load carrying capacity of masonry arches

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ABSTRACT: The standard assessment procedures for masonry arches assume the compressive strength of masonry as a deterministic variable. In this paper, the load carrying capacity of masonry arches is addressed by means of Kinematic Limit Analysis (Mechanism Method) assuming the material compressive strength as a random variable. Based on the failure probability of each admissible mechanism, the global collapse probability is provided by the Cornell approximation. The application of the procedure to a single span bridge shows that, for arch-type structures, the C.o.V. of the load carrying capacity is less than the C.o.V. of the compressive strength. Discussion is provided on the safety factors defined by masonry codes and their effect on the safety assessment of masonry arches.

1 INTRODUCTION

Several procedures have been proposed for the safety assessment of arch-type structures, among which masonry bridges are far the most important, some based on the Limit Analysis Theorems (Heyman 1982, Crisfield 1985, Crisfield and Packham 1988, Harvey 1988, Hughes and Blackler 1997, Cavicchi and Gambarotta 2005, Gilbert 2005), others on non linear incremental procedures (Crisfield 1985, Bridle and Hughes 1990, Hughes and Blackler 1997, Molins and Roca 1998, Brencich and De Francesco 2004, among the others). A common assumption is that the compressive strength attains the same deterministic value in every section of the arch. The C.o.V. of existing structures, that may be as large as 25% (Page, 1981, Ellingwood and Tallin 1985, Dymiotis and Gutlederer 2002, Brencich and Morbiducci 2007), suggests that it should be considered a random variable. The code-type approaches assume reduced values of the design strength in order to take into account the material randomness.

In this paper, the compressive strength of masonry is assumed as a random variable; the other sources of uncertainty, such as the geometry of the arch, the interaction between structural elements, the mechanical response of the interfaces and the variability of the live loads are assumed deterministic. As a consequence of the material strength randomness, the axial thrust-bending moment limit domain, to which the safety assessment of masonry bridges is often related, is random too. The Load Carrying Capacity (LCC) of the arch is calculated by means of Kinematic Limit Analysis (Mechanism Method) considering the energy dissipation due to both the bending moment and the axial thrust.

A conservative estimate of the global failure probability of the arch is derived from the failure probability of each admissible mechanism on the basis of the Cornell approximation (Cornell 1967). The comparison of the results with the classical safety format of modern codes shows that the standard safety coefficients are reasonable values if the knowledge of masonry is limited and standard stochastic distributions of the material properties are assumed. If the probabilistic description of the material could be enhanced, for example through a wide testing campaign, reduced safety factors could be assumed.

2 PROBABILISTIC LIMIT ANALYSIS OF ARCHES

The Limit Analysis approach to the assessment of an arch assumes a collapse mechanism for the arch due to the activation of four hinges in the arch. At the Ultimate Limit State (ULS), the collapse load is computed equating the work done by the external forces and the work by the internal actions, i.e. by the plastic moments in the hinges. Usually, the contribution of the axial thrust to the internal work is neglected (Heyman, 1982). In this frame, the constitutive model for the material needs to be that of an elastic-perfectly plastic material with no limit to the inelastic strains.

In order to take into account also the contribution of the axial thrust to the internal work, at the ULS both the plastic moment and the related axial thrust transmitted through the section need to be considered as dissipating energy on the relative rotations and axial displacements respectively. To this aim, two modifications to the standard approach need to be introduced: i) the limit domain needs to be defined in the axial thrust-bending moment plane through some probabilistic criterion; ii) the kinematics of the collapse mechanism needs to take into account also the axial displacements.

2.1 Probabilistic limit domain and plastic hinges

Masonry is modelled as a No-Tensile-Resistant (NTR) material with a perfectly-plastic compressive response; this latter assumption is not ideal but is unavoidable in the frame of Limit Analysis and further discussion (Brencich and Gambarotta 2005, Brencich et al. 2007) would be beyond the scope of this paper. With the aim of describing the stress distribution of the cross section through the internal forces (axial thrust N - bending moment M), the limit domain is defined in the N-M plane as a parabola, Eq. (1):

$$\phi(N,M)=\frac{1}{2}\left(\frac{N}{bhf_c}\right)\left[1+\left(\frac{N}{bhf_c}\right)\right]+\frac{|M|}{bh^2 f_c}=0 \tag{1}$$

where b and h are the base and height of the cross section and $\tilde{f}_c$ the material compressive strength assumed as random over the arch but constant on the cross section; the N-M limit domain of Eq. (1) turns out to be not unique, depending on the value attained by the material strength section by section. The probability of failure of the arch section is the probability that the compressive strength does not exceed a threshold value in that section:

$$\mathrm{P}_f=\mathrm{P}\left(\tilde{f}_c\le-\frac{N}{bh}\left(\frac{Nh}{Nh+2|M|}\right)\right). \tag{2}$$

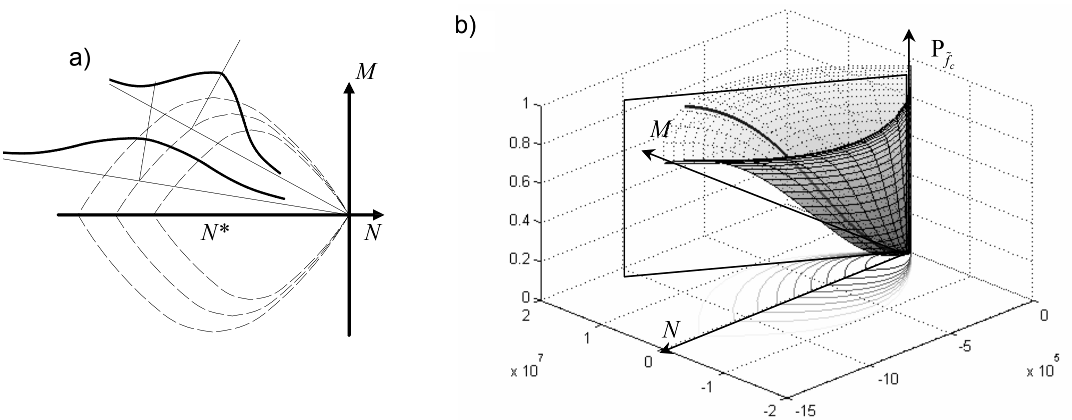

Figure 1. Limit domains and their probability a) PDF; b) CDF. Limit domains on the horizontal plane.

The meaning of Eq. (2) is represented in Fig. 1: in each cross section the limit domain can be defined by means of a mean value and a distribution function, Fig. 1.a, showing every domain is possible on principle but with different probability density (PDF); the cumulative probability (CDF) is displayed on vertical axis of Fig. 1.b.

2.2 The kinematic model

The internal dissipation of masonry arches is due to the deformation rates (velocities) of the displacements, rather than to the displacements, associated to the bending moment and the axial thrust, i.e. to the deformation rates of the relative rotation between the cross section faces and the axial displacement. To this aim, the arch is reduced to its axis and is described by means of a finite number of sections (nodes) on the axis. The nodes are potential plastic hinges, so that the dissipation and the deformation rates of the arch are concentrated in a finite number of sections, while the parts in-between two successive nodes are assumed to be rigid. The active forces are: i) the dead loads (arch and fill) reduced to the arch axis ($\mathbf{f}_s$ and $\mathbf{f}_f$ respectively, $\mathbf{f}_0 = \mathbf{f}_s + \mathbf{f}_f$); ii) the live loads $\boldsymbol{p}_s$ applied to the arch line; s_k is the multiplier of the live loads only.

At collapse, the external power W_{ext} equals the internal dissipated energy D_{int}:

$$W_{ext} = s_k \int \mathbf{f}_a^T \dot{\mathbf{u}} ds + \int \mathbf{f}_0^T \dot{\mathbf{u}} ds = D_{int} \geq 0\,, \tag{3}$$

where $\dot{\mathbf{u}}$ is the global vector of the nodal velocities. The strain rates of the *i-th* node $\dot{\mathbf{u}}_i=\{\dot{u}_i, \dot{v}_i, \dot{\varphi}_i\}^T$, related to a global reference system, Fig. 2.a, can be expressed in terms of the relative displacement rates (axial displacement and relative rotation in node *i*) $\dot{\boldsymbol{\varepsilon}}_i=\{\dot{w}_i, \dot{\theta}_i\}^T$, of the previous *i-1* nodes, Fig. 2.b:

$$\dot{u}_i = -\sum_{k=1}^{i-1} \dot{\vartheta}_k \sum_{j=k}^{i-1} l_j^y + \sum_{k=1}^{i} \dot{w}_k \cos(\beta_k)\,,\quad \dot{v}_i = -\sum_{k=1}^{i-1} \dot{\vartheta}_k \sum_{j=k}^{i-1} l_j^x - \sum_{k=1}^{i} \dot{w}_k \sin(\beta_k)\,,\quad \dot{\varphi}_i = \sum_{k=1}^{i} \dot{\vartheta}_k \tag{4}$$

being β_k the orientation of the *k-th* arch segment in-between nodes *i-1* and *i*; l_j^x and l_j^y geometric quantities defined in Fig. 2.a. The global vectors $\dot{\mathbf{u}}$ and $\dot{\boldsymbol{\varepsilon}}$ are obtained from the correspondent nodal vectors $\dot{\mathbf{u}}_i$ and $\dot{\boldsymbol{\varepsilon}}_i$ by proper ordering; relating Eq. (4) to the global vectors we obtain:

$$\dot{\mathbf{u}} = \mathbf{A}\, \dot{\boldsymbol{\varepsilon}}\,, \tag{5}$$

where matrix **A** depends only on geometric parameters (arch geometry and number of nodes).

Vector $\dot{\boldsymbol{\varepsilon}}_i$, collecting the relative displacement rates at the *i-th* section, is obtained assuming an associate plastic flow rule, so that $\dot{\boldsymbol{\varepsilon}}_i$ is proportional to the outward normal $\boldsymbol{n}_i=\{d_x, d_y\}_i^T$ to the limit domain via the plastic multiplier λ_i:

$$\dot{\boldsymbol{\varepsilon}}_i=\{\dot{w}_i, \dot{\theta}_i\}^T = \lambda_i\, \boldsymbol{n}_i = \lambda_i\, \{d_x, d_y\}_i^T\,. \tag{6}$$

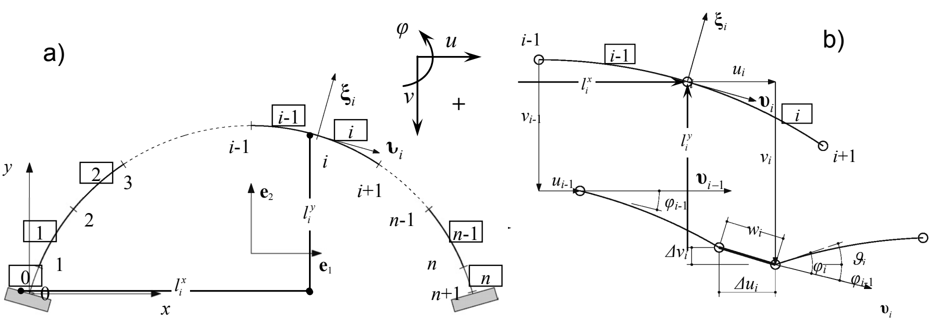

Figure 2. a) The arch model; b) relative displacement rates (axial displacement and rotation) at a node. Strain rates are indicated through finite increments instead of upper dots.

Eq. (6) allows Eq. (5) to be rewritten as:

$$\dot{\mathbf{u}} = \mathbf{A}\,\mathbf{D}\,\boldsymbol{\lambda}. \tag{7}$$

where vector $\boldsymbol{\lambda}$ collects the plastic multipliers of the four plastic hinges and matrix **D** contains the components of the outward normal vectors to the limit domain.

It can be shown that the internal dissipated power D_{int} of Eq. (2) depends only on the distance of the point on limit domain line from the origin of the *N-M* plane; the external power W_{ext} and the internal one D_{int} can thus be expressed as:

$$W_{ext} = (\mathbf{f}_0 + s\,\mathbf{f}_a)^T\,\dot{\mathbf{u}}\ ; \quad D_{int} = \mathbf{r}^T\boldsymbol{\lambda}\ , \tag{8}$$

being **r** a vector collecting the distance of the limit domain sides from the origin, figure 3.a.

The collapse mechanism is due to four plastic hinges, *h*, *k*, *l* and *m*, figure 3.b, each one located on some part of the (linearized) limit domain, Fig. 3.a; let us say that hinges *h*, *k*, *l* and *m* are on the segments *p*, *q*, *r* and *s* of the domain, respectively. In order to take into account all the possible mechanisms, the hinges need to be located in every node, avoiding repetitions, and each hinge, once the location is defined, can be on any one of the *2b* segments of the limit domain. Such a mechanism will be addressed with the subscripts *hklm* and superscripts *pqrs* respectively. For an arch with *n* nodes, the number of geometrically admissible mechanisms $N_{geom\text{-}adm}$ is given as:

$$N_{geom\text{-}adm} = [\ n! * (2b)^4\ ] \ /\ [\ 4! * (n\text{-}4)!\ ]. \tag{9}$$

Kinematically admissible mechanisms are a subset V_n of the geometrically admissible mechanisms of Eq. (9) since they need to have positive external power ($W_{ext} > 0$). The actual number $N_{mech\text{-}adm}$ of kinematically admissible mechanisms cannot be estimated *a priori*, so that the entire set of geometrically admissible mechanisms of eq. (9) needs to be scanned case by case to find out the kinematically admissible ones (Sterpi 2006).

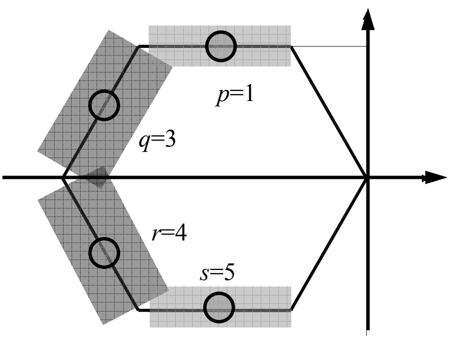

Figure 3.a Position of the hinge on the domain (p=1, q=3, r=4, s=5)

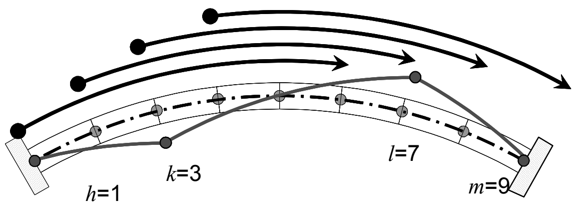

Figure 3.b Position on the arch of the 4 hinges (h=1, k=3, l=7, m=9)

2.3 *The probabilistic model*

For a given kinematically admissible ($W_{ext} > 0$) displacement vector $\dot{\mathbf{u}}^{pqrs}_{hklm}$ the external power and the internal dissipated energy are given by Eq. (3) set equal to zero:

$$\mathbf{u}^{pqrs}_{hklm} \in V_n : W_{ext} = \left(\mathbf{f}_0 + s^{pqrs}_{hklm}\mathbf{f}_a\right)^T \dot{\mathbf{u}}^{pqrs}_{hklm} = D_{int} = \mathbf{r}^T\boldsymbol{\lambda}^{pqrs}_{hklm}\,, \tag{10}$$

leading to the collapse multiplier as:

$$s^{pqrs}_{hklm} = \left(\mathbf{r}^T\boldsymbol{\lambda}^{pqrs}_{hklm} - \mathbf{f}_0^{\,T}\mathbf{u}^{pqrs}_{hklm}\right) \Big/ \left(\mathbf{f}_a^{\,T}\dot{\mathbf{u}}^{pqrs}_{hklm}\right); \tag{11}$$

subscripts and superscripts indicate that the multiplier has been deduced for a specific collapse mechanism. The material compressive strength is contained in the vector **r**; vector λ^{pqrs}_{hklm} has only four non-zero components, the ones related to the active plastic hinges, while all the other components, related to the nodes where no plastic hinge is active, are zero.

Every collapse mechanism belonging to the set V_n can be considered as a possible event of collapse E_i. Since several collapse mechanisms are possible, the actual failure probability P_f is derived summing up the probability associated to each mechanism and adding/deducing the joint probabilities:

$$P_f = P(E) = P\,[E_1 + E_2 + E_3 + \ldots..] = P(E_1) + P(E_2) + P(E_3) + \ldots\ldots - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E3) \ldots + P(E_1 \cap E_2 \cap E_3) + \ldots \quad (12)$$

Eq. (12) is difficult to be estimated due to the joint probabilities and can be approximated by the two Cornell limits (Cornell 1967):

$$\max_{i=1}^{N_{mech}} \{P(E_i)\} \le P_f \le \sum_{i=1}^{N_{mech}} P(E_i). \quad (13)$$

We now assume for the random variable $\tilde{f}_c$ a Gaussian distribution, according to standard code-type approaches (Ditlevsen 1979, Ditlevsen 1983, Ditlevsen and Bjerager 1986). Due to the stability of the Gaussian distribution (Augusti et al. 1984, Elishakoff 1999) the average value of the collapse multiplier is derived from Eq. (11) assuming the vector **r** calculated for the average value $\tilde{f}_c$, while the coefficient of variation (C.o.V.) is given as:

$$\text{C.o.V.}\left(s_{hklm}^{pqrs}\right) = \sqrt{\left(\boldsymbol{\lambda}_{hklm}^{pqrs}\right)^T \mathbf{C}(\mathbf{r})\,\boldsymbol{\lambda}_{hklm}^{pqrs}} \Big/ \left(\mathbf{r}^T \boldsymbol{\lambda}_{hklm}^{pqrs} - \mathbf{f}_0^T \mathbf{A}\mathbf{D}\boldsymbol{\lambda}_{hklm}^{pqrs}\right), \quad (14)$$

where **C**(**r**) is the co-variance matrix of **r** (for independent random variables it is simply a diagonal matrix with the variances of **r**). Eq. (14) shows that the C.o.V. of the multiplyer differs from the C.o.V. of the compressive strength: the structure plays the role of a filter to the material randomness. Once the average and C.o.V. values of the Gaussian distribution of the collapse multiplier is known, also its cumulative probability function is known.

3 APPLICATION TO A REAL CASE: THE *PRAROLO* BRIDGE

The *Prarolo* bridge is an in service single-arch two-rails bridge on the Genoa-Turin line in north-western Italy, built in 1851-1852, Fig. 4, spans 40m over the Scrivia River (rise: 10m). The skewbacks are built on approx. cylindrical piers (diameter: 20m, height: 22m) assumed as perfect constraints. The arch thickness is 3m close to the springing and 1.8m in crown; the fill height in crown is 1.25m, and the whole bridge deck is 5m wide. Brickwork density is assumed 18 kN/m^3, for the fill: 16 kN/m^3. The arch barrel is simplified assuming a constant average thickness of 2.4m, leading to an average radius of 29.7m and a rise corrected to 10.10m.

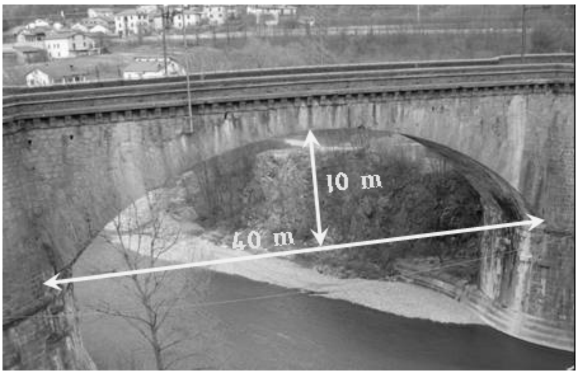

Figure 4. The *Prarolo* bridge.

The bridge model is divided into 9 nodes and the limit domain is linearized with ten straight segments, Fig. 5, assuming an average compressive strength of 10MPa and C.o.V.s of 10% and 20%, the latter being a quite standard value while the first one is too optimistic for a brickwork structure. Grey areas show the standard deviation for the different straight segment of the limit domain. The number of collapse mechanisms as calculated by using Eq. (9) is 1.26 10^6, 75625 of which are kinematically admissible ($W_{ext} > 0$) ones (6% of the total number). The load is the D4-type carriage, i.e. 225 kN/axle (UIC 1995), figure 5.a, s being the load multiplier.

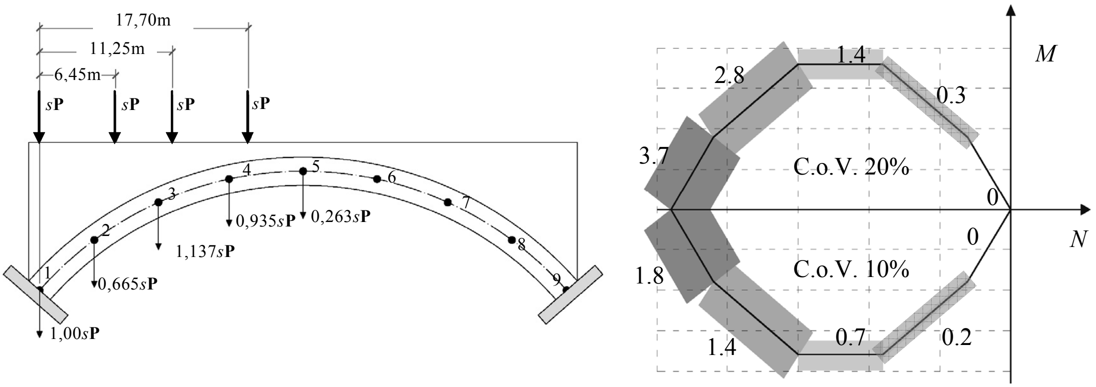

Figure 5. a) Reduced geometry of the bridge + scheme of the D4 carriage (225kN/axle - 4axles); b) limit domain with a 20% and 10% C.o.V. Numbers indicate the variance of the compressive strength in MPa for each part of the domain. f_c = 10 MPa

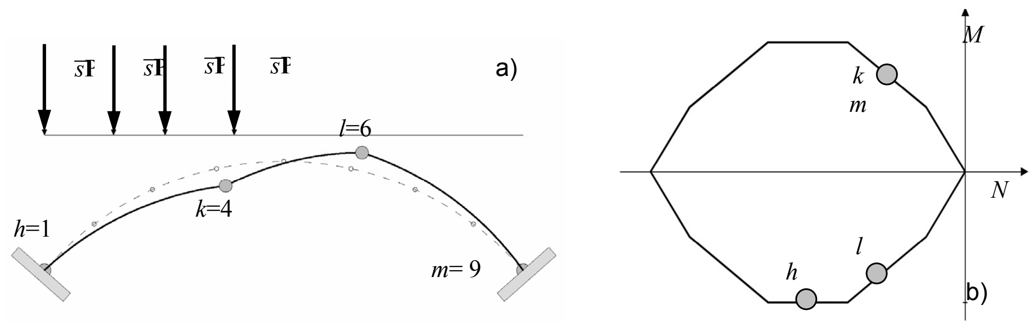

Figure 6. a) Collapse mechanism (: *h*=1, *k*=4, *l*=6, *m*=9; b) location of the hinges on the limit domain.

Fig. 6 shows the mechanism with the lowest collapse multiplier and the position of the plastic hinges on the limit domain. Fig.s 7.a and 7.b show the cumulative distribution function of the load multipliers of the collapse mechanisms with lower multipliers while in Fig.s 7.c and 7.d are enlargements of the diagrams close to the failure probability of 10^{-3}. This threshold is substantially coherent with the code-type approaches: the global failure probability is approximately 10^{-6}, but the assumption of deterministic loads increases it to 10^{-3} (Augusti et al. 1984). The specific values of these thresholds are inessential for the procedure herein discussed and the value of 10^{-6} and 10^{-3} are notional values that should not be taken as standards without specific risk analyses. Dashed lines of figure 9 are the upper and lower bounds of Eq. (13).

Table 1 compares: i) the load for which the arch has a failure probability equal to the assigned threshold (10^{-3}) on the Upper and Lower Bound curves (columns U.B. and L.B., the assumed ultimate load is the Upper Bound estimate); ii) according to masonry design codes (ENV 1998, IT. Nat. Code 1987), the design compressive strength is derived from the characteristic value (5% fractile) by a safety factor equal to 1 (columns 4-5), 2.2 (ENV 1998, columns 6-7) and 3 (IT Nat. Code, columns 8-9) and the ultimate load is estimated by means of a deterministic model.

The estimated limit loads, Table 1, are in partial agreement with design codes: if we assume a 20% C.o.V. for the masonry compressive strength, the probabilistic approach estimates the collapse load to be 412 kN/axle, while EuroCode 6 would lead to an estimate of 563 kN/axle and the Italian National Code to a highly conservative estimate of 340 kN/axle. For a C.o.V. of 10%, which is probably an optimistic value on which the safety factors of the codes have not been calibrated, Table 1 shows that the code-type estimates would be highly conservative if compared to a fully probabilistic approach.

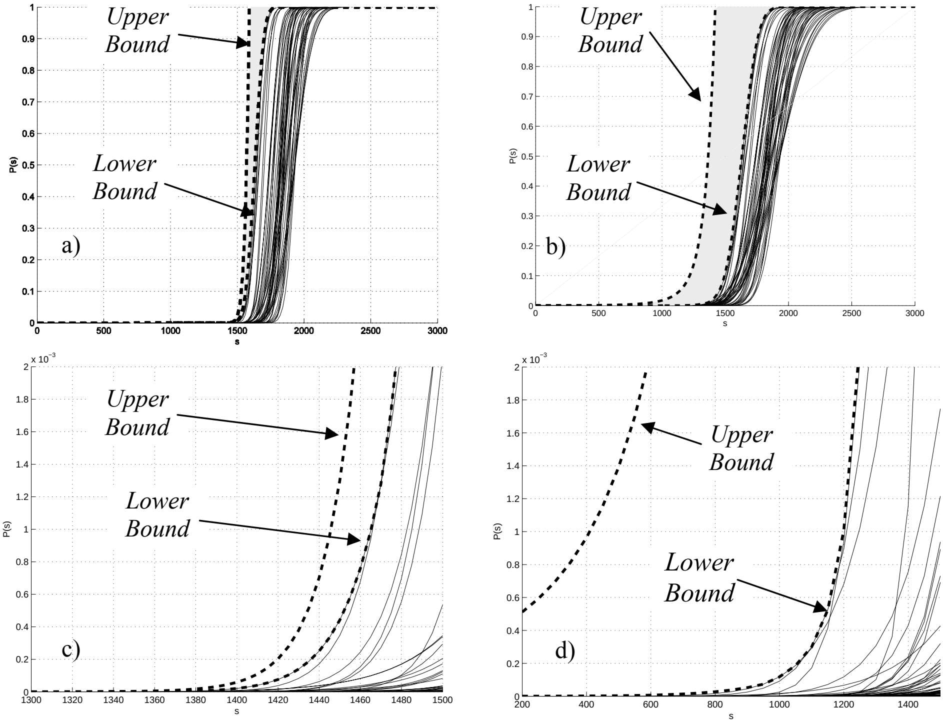

Figure 7. Cumulative distribution functions of the failure probability for a C.o.V. of: a) 10%; b) 20% of the masonry compressive strength; c) and d) are zooms of the diagrams close to the failure probability of 10^{-3}.

Table 1. Upper Bounds (UB) and Lower Bounds (LB) from probabilistic analysis and semi-probabilistic values (code approach). Axle loads.

	UB [kN]	LB [kN]	$f_{cd} = f_{ck}$	P_{coll} [kN]	$f_{cd} = f_{ck}/2.2$	P_{coll} [kN]	$f_{cd} = f_{ck}/3$	P_{coll} [kN]
c.o.v. 10%	1446	1465	8.4	1465	3.8	740	2.8	500
c.o.v. 20%	412	1200	6.7	1280	3.0	563	2.2	340

4 CONCLUSIONS

The procedure discussed in this paper is an enhancement to the Kinematic Limit Analysis approach (Mechanism Method) commonly accepted for the assessment of arches. In the frame of this approach, a full probabilistic analysis, though with some simplifying assumption, has been discussed aiming at outlining the effect of material inhomogeneity (compressive strength) on the ultimate load. Present masonry design codes do not deal explicitly with this issue but implicitly assume partial safety coefficients that are expected to include it.

The results provided by the procedure applied on a prototype derived from a real bridge show that the code-type approach is somehow calibrated on materials with high dispersion of the mechanical parameters (C.o.V. = 20%). Reduced C.o.V.s, that can be obtained by means of wide experimental campaigns, would allow lower safety coefficients. A more refined procedure should better estimate the Cornell Limits, since figure 9.c and 9.d show that the Upper and Lower Bounds may be quite distant: the assumption of the Upper Bound as the failure probability curve implicitly introduces a further safety coefficient that remains unknown for the present state of knowledge and should be better estimated. These results are not unexpected but still need to be investigated before design codes can introduce it into design rules.

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